§11. 坐标系与坐标架.

坐标系 $\overset{\partial}{\rightsquigarrow}$ 坐标架

$$\begin{tikzcd} S \arrow[r] & TS \cong S\times \mathbb{R}^2 \arrow[d] \\ & S \end{tikzcd}$$

, TS是S的一阶近似.

令S为曲面. 设$p\in S$. 我们知道

事实一: 在p的某一邻域, 存在S上的正交坐标系 (正交参数系)

即 $\exists (u,v) \xrightarrow{r} S$, s.t

$$F=\langle r_u, r_v\rangle = 0.$$

但是:

事实二: 在p的某一邻域, 存在S上的单位正交坐标系.

$\Longleftrightarrow$ S在p点附近与平面等距同构.

特别得, S在p点附近, 高斯曲率 $\equiv 0$.

所以, 一般曲面上不存在单位正交坐标系. 能做到的, 是如下

事实三: $\forall p\in S$, 必定存在某个邻域, 使得与平面保角同构. 即

存在等温坐标系. $\left(\mathrm{I}(u,v)=\lambda(u,v)(du^2+dv^2)\right)$

此时: 不仅 $F=0$, 且 $E=G$.

但是对于坐标架就不同了：

设 $\{r_u, r_v\}$ 为自然标架，则通过 Gram-Schmidt 正交化过程可得单位正交标架

$$\left\{ e_1 = \frac{r_u}{|r_u|}, \quad e_2 = \frac{r_v - \langle r_u, e_1\rangle e_1}{|r_v - \langle r_u, e_1\rangle e_1|} \right\}$$

设 $\theta = \theta(u,v)$，则若 $\{e_1, e_2\}$ 为单位正交标架，则

$$\{ \tilde{e}_1 = \cos\theta\, e_1 + \sin\theta\, e_2, \quad \tilde{e}_2 = -\sin\theta\, e_1 + \cos\theta\, e_2 \}$$

亦为单位正交标架。

所以：曲面上存在（很多）单位正交标架。但

问题：我们可以用单位正交标架做几何吗？

令 $\{e_1, e_2\}$ 为 S 上的单位正交标架，令 $e_3 = e_1 \times e_2$，则 $\{r; e_1, e_2, e_3\}$ 构成 S 上的正交标架场。

令 $\omega_1 = \langle dr, e_1\rangle$，$\omega_2 = \langle dr, e_2\rangle$。则

$$I = \langle dr, dr\rangle = \langle \omega_1 e_1 + \omega_2 e_2, \omega_1 e_1 + \omega_2 e_2\rangle = \omega_1^2 + \omega_2^2$$

「注意到：$\tilde{\omega}_1 = \omega_1\cos\theta + \omega_2\sin\theta$，$\tilde{\omega}_2 = -\sin\theta\,\omega_1 + \cos\theta\,\omega_2$，则

$$\tilde{\omega}_1^2 + \tilde{\omega}_2^2 = \omega_1^2 + \omega_2^2$$」

令 $\omega_{13} = \langle dn, e_1\rangle$，$\omega_{23} = \langle -dn, e_2\rangle$

则 $\mathrm{II} = -\langle dr, dn\rangle = \langle \omega_1 e_1+\omega_2 e_2,\ \omega_{13}e_1+\omega_{23}e_2\rangle$

$$= \omega_1\cdot\omega_{13}+\omega_2\cdot\omega_{23}$$

「也注意到

$$\tilde\omega_{13} = -\langle dn, \tilde e_1\rangle = -\langle dn, \cos\theta\, e_1+\sin\theta\, e_2\rangle$$

$$= \cos\theta\cdot\omega_{13}+\sin\theta\,\omega_{23}$$

$$\tilde\omega_{23} = -\sin\theta\,\omega_{13}+\cos\theta\cdot\omega_{23}$$

$$\Rightarrow \tilde\omega_1\cdot\tilde\omega_{13}+\tilde\omega_2\tilde\omega_{23} = \omega_1\cdot\omega_{13}+\omega_2\omega_{23}$$ 」

$$\mathrm{III} = \langle dn, dn\rangle = \omega_{13}^2+\omega_{23}^2$$

「 $K\cdot\mathrm{I} - 2H\,\mathrm{II} + \mathrm{III} = 0$ 」

~~一般[illegible]~~

$\cancel{\psi = \omega_1\omega_{23}-\omega_2\omega_{13}}$

运动方程：

$$\begin{cases} dr = \omega_1 e_1+\omega_2 e_2 \\ de_1 = \omega_{12}e_2+\omega_{13}e_3 \\ de_2 = \omega_{21}e_1+\omega_{23}e_3 \\ de_3 = \omega_{31}e_1+\omega_{32}e_2 \end{cases} \qquad \begin{cases} \omega_{21} = -\omega_{12} \\ \omega_{31} = -\omega_{13} \\ \omega_{32} = -\omega_{23} \end{cases}$$

$\mathcal{O} = \{u, v\}$ 为变量的 C^∞ 函数 $= \{f = f(u,v)\}$

$\Omega^1 = \{$一阶外微分形式$\} = \mathcal{O}\cdot\{du, dv\} = \{ f\,du + g\,dv \mid f, g \in \mathcal{O}\}$

$\Omega^2 = \{$二阶外微分形式$\} = \mathcal{O}\{du \wedge dv\} = \{f\,du\wedge dv \mid f \in \mathcal{O}\}$

$$\mathcal{O} \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \triangleq \Lambda^2\Omega^1 \qquad \Omega^1 \otimes_{\mathcal{O}} \Omega^1 \xrightarrow{\Lambda^2} \Omega^2$$

$$d_0(f) = d(f) \triangleq \frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv$$

$$d_1(f\,du + g\,dv) \triangleq d_0(f)\wedge du + d_0(g)\wedge dv$$

(1) Λ^2 $\mathcal{O}$-linear

(2) $\theta\wedge\psi = -\psi\wedge\theta$

$\Leftrightarrow \theta\wedge\theta = 0$

$$= \left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv\right)\wedge du$$

$$+ \left(\frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv\right)\wedge dv$$

$$= \left(\frac{\partial f}{\partial v} + \frac{\partial g}{\partial u} \right) du\wedge dv$$

注意到 $d^2(f) = d_1 \cdot d_0(f)$

$$= d_1\left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv\right)$$

$$= \left(\frac{\partial^2 f}{\partial u \partial v} - \frac{\partial^2 f}{\partial v \partial u}\right) du\wedge dv$$

$$= 0$$

命题: (i) $\omega_1\wedge\omega_2 = \sqrt{EG-F^2}\, du\wedge dv$　曲面的有向面积元

(ii) $\omega_{13}\wedge\omega_{23} = K\cdot\omega_1\wedge\omega_2$　~~高斯映射~~

证明：取 $\{u, v\}$ 为正交坐标系，则

取 $e_1 = \dfrac{r_u}{|r_u|}$，$e_2 = \dfrac{r_v}{|r_v|}$

则 $w_1 = \sqrt{E}\,du$，$w_2 = \sqrt{G}\,dv$

$$\begin{cases} dw_1 = w_{12}\wedge w_2 \\ dw_2 = w_{21}\wedge w_1 = -w_{12}\wedge w_1 \end{cases}$$

设 $w_{12} = a\,du + b\,dv$

$\Rightarrow a\sqrt{G}\,du\wedge dv = -(\sqrt{E})_v\,du\wedge dv$

$\Rightarrow a = \dfrac{-(\sqrt{E})_v}{\sqrt{G}}$

同理 $b = \dfrac{(\sqrt{G})_u}{\sqrt{E}}$.

$w_1\wedge w_2 = \sqrt{EG}\,du\wedge dv$

故 $dw_{12} = -K\,w_1\wedge w_2 = -K\cdot\sqrt{EG}\,du\wedge dv$

$$\Rightarrow d\left(\frac{(-\sqrt{E})_v}{\sqrt{G}}du + \frac{(\sqrt{G})_u}{\sqrt{E}}du\right) = -K\cdot\sqrt{EG}\,du\wedge dv$$

$$\| $$

$$\left[\left(\frac{-(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right]du\wedge dv$$

证明:

(i) $\{\omega_1, \omega_2\} = \{du, dv\}\cdot A = \{du, dv\}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$A\cdot A^T = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$\Rightarrow \omega_1\wedge\omega_2 = \det A\cdot du\wedge dv$$

$$= \sqrt{EG-F^2}\, du\wedge dv$$

是

(ii) $\{\omega_{13}, \omega_{23}\} = \{\omega_1, \omega_2\}\cdot B.$ $\begin{cases} \det B = K \\ \mathrm{tr} B = 2H \end{cases}$

$$\omega_{13}\wedge\omega_{23} = \det B\cdot d\omega_1\wedge\omega_2$$

$$= K\, d\omega_1\wedge\omega_2.$$

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正交标架下的结构方程.

$$d^2(r) = d^2(e_i) = 0 \Rightarrow$$

$$\begin{cases} \begin{cases} d\omega_1 = \omega_2\wedge\omega_{21} \\ d\omega_2 = \omega_1\wedge\omega_{12} \end{cases} & \rightsquigarrow \text{联络} \\ \begin{cases} d\omega_{12} = \omega_{13}\wedge\omega_{32} \\ \quad\;\; = -K\omega_1\wedge\omega_2 \end{cases} & \rightsquigarrow \text{Gauss 方程} \\ \begin{cases} d\omega_{13} = \omega_{12}\wedge\omega_{23} \\ d\omega_{23} = \omega_{21}\wedge\omega_{13} \end{cases} & \rightsquigarrow \text{Codazzi 方程.} \end{cases}$$

命题: (i) $d\omega_{12} = -K\omega_1\wedge\omega_2$ 表达 Gauss 方程

(ii) $\begin{cases} d\omega_{13} = \omega_{12}\wedge\omega_{23} \\ d\omega_{23} = \omega_{21}\wedge\omega_{13} \end{cases}$ 表达 Codazzi 方程

$$\Rightarrow \quad K = -\frac{1}{\sqrt{EG}}\left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right] \qquad \text{Gauß方程}$$

用这验证 (ii).

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